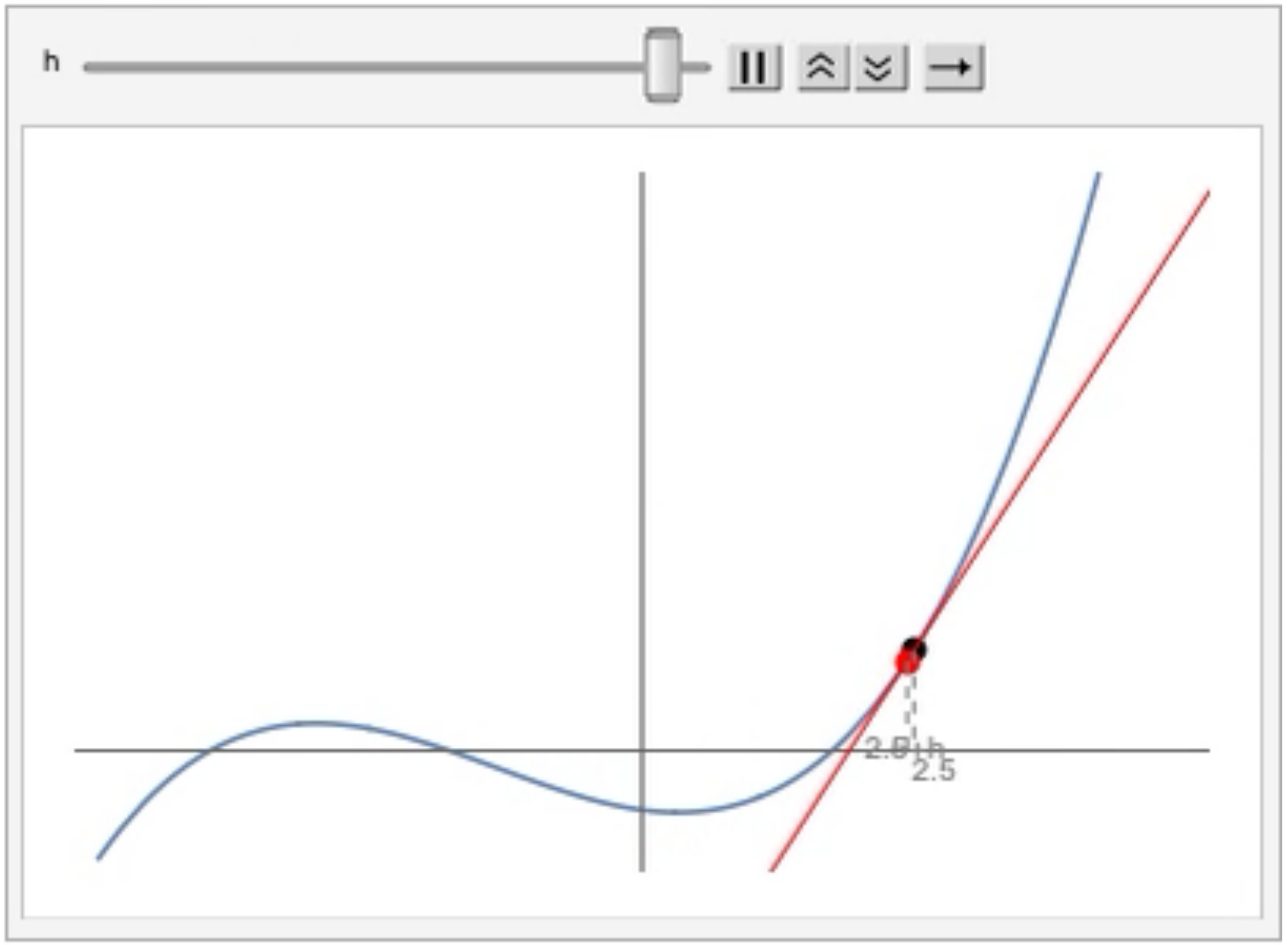
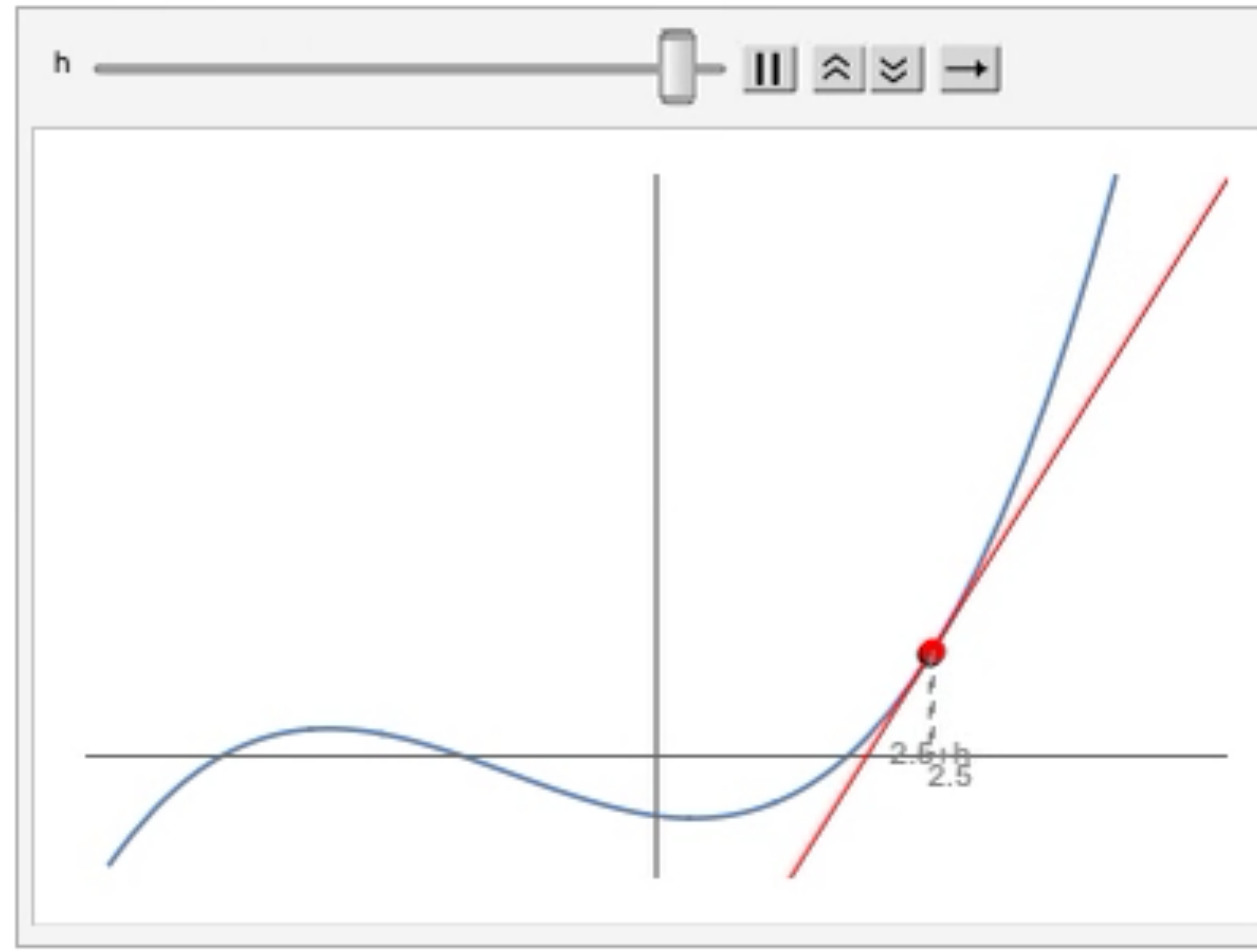
Intro Video: section 2.7
Derivatives and rates of change

Math F251X: Calculus I

How can we find the slope of a tangent line?

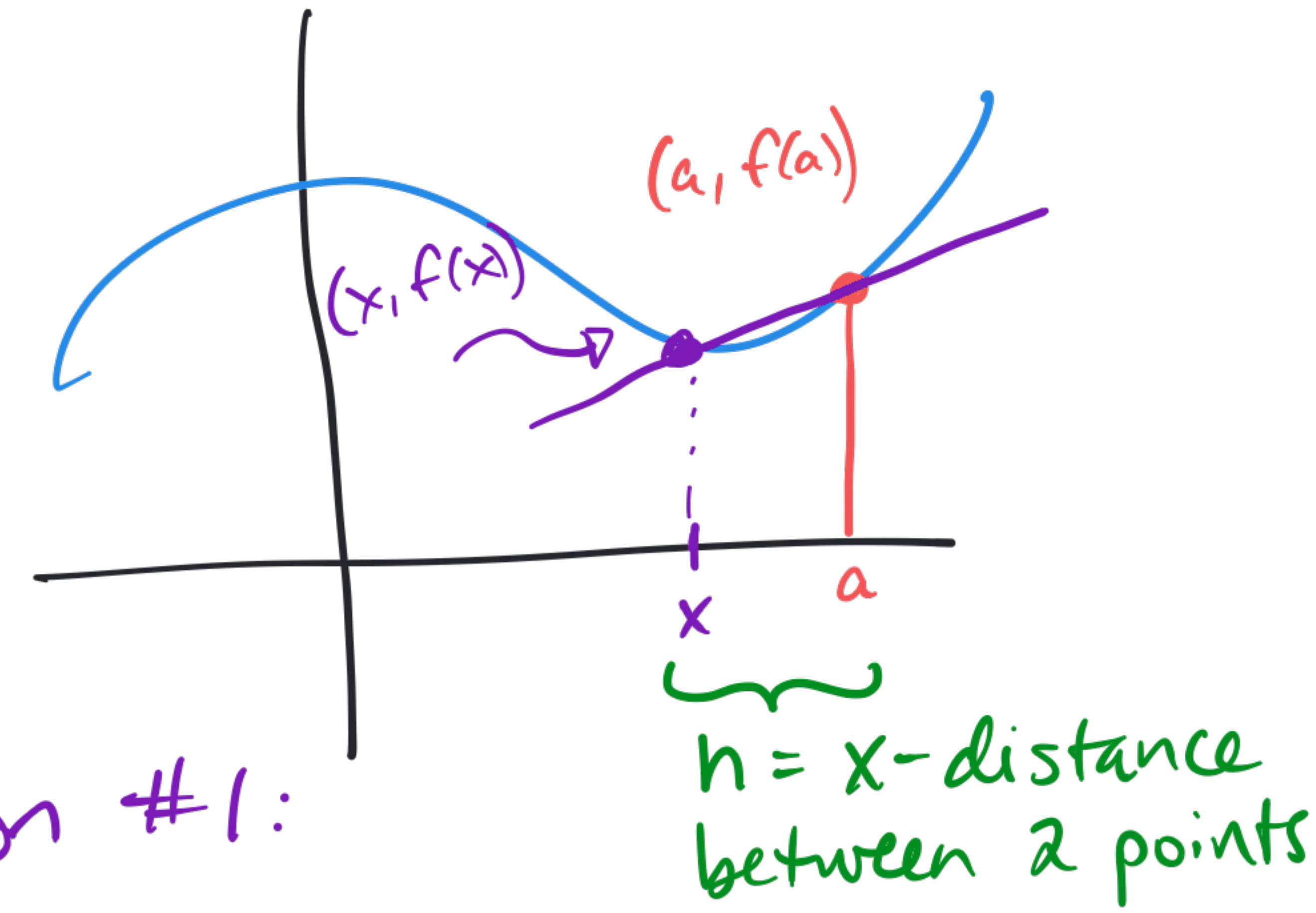




Determining the slope of a tangent line, exactly!

Slope of secant line $= \dfrac{\Delta y}{\Delta x}$

$$= \frac{f(x) - f(a)}{x - a}$$



Slope of tangent line, definition #1:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \text{slope of tangent line at } (a, f(a))$$

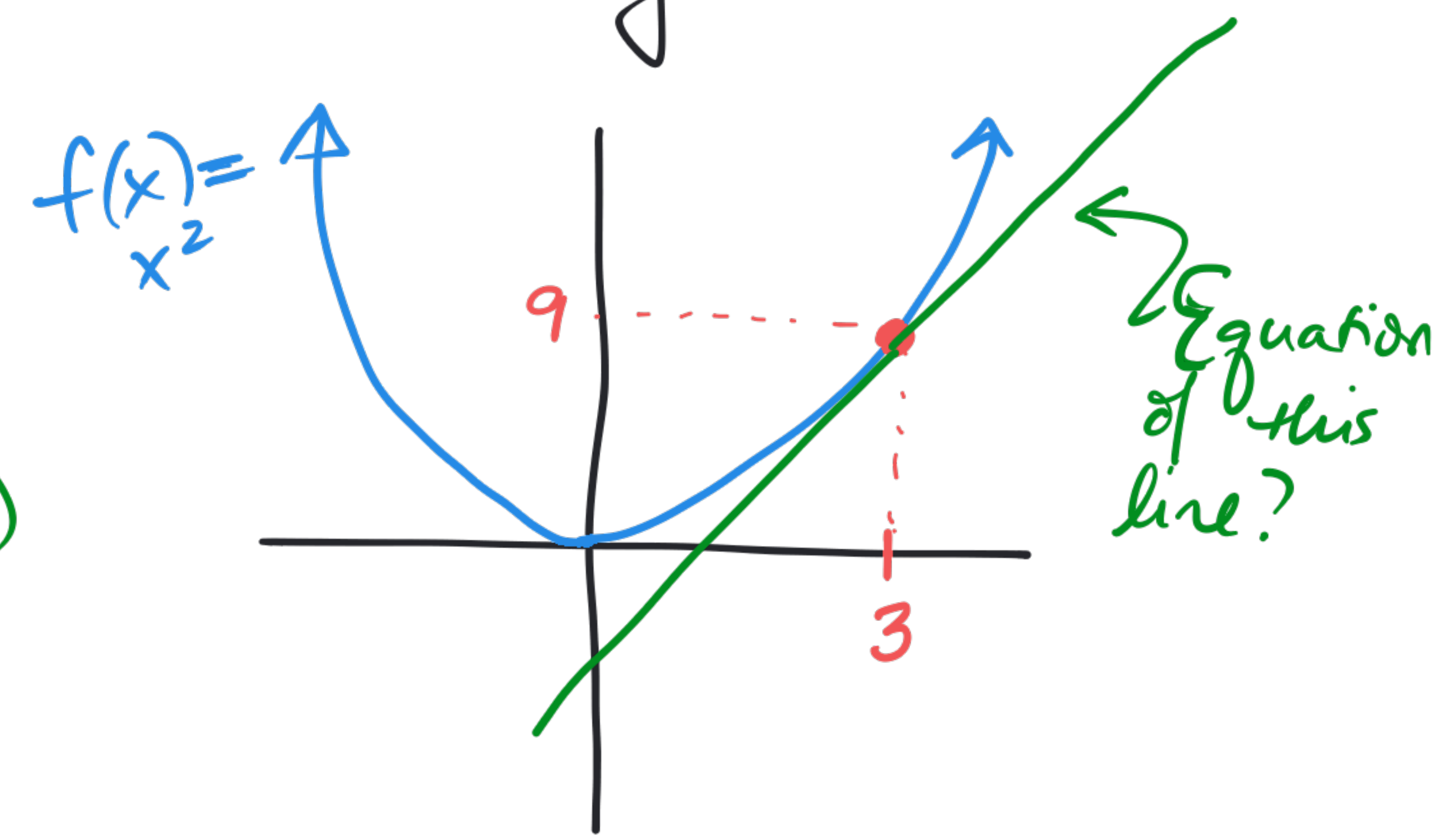
Slope of tangent line, definition #2:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \text{slope of TL at } (a, f(a))$$

The <u>derivative</u> of f at a

Example: Write the equation of the tangent line at the point $(3, 9)$ to the curve $y = x^2$.

Point on TL: $(3, 9)$



Slope of TL at $x = 3$ $= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$

$= \lim_{x \to 3} \frac{x^2 - 3^2}{x - 3}$

$= \lim_{x \to 3} \frac{\cancel{(x - 3)}(x+3)}{\cancel{x - 3}} = \lim_{x \to 3} (x+3) = 6$

Tangent line has equation $\boxed{y = 6(x - 3) + 9}$

TL to $f(x) = x^2$ at $(3,9)$, definition #2

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h}$$

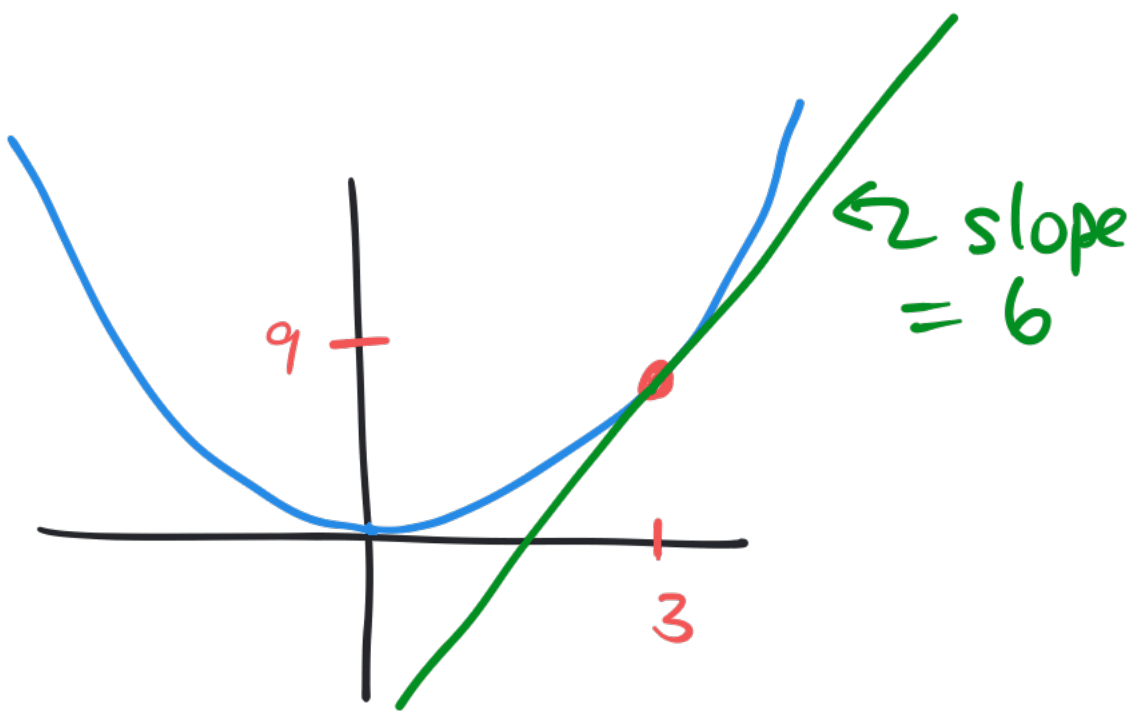
$$= \lim_{h \to 0} \frac{1}{h}\left((3+h)^2 - 3^2\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\cancel{3^2} + 6h + h^2 - \cancel{3^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(6\cancel{h} + h^{\cancel{2}}\right)$$

$$= \lim_{h \to 0} 6 + h$$

$$= 6$$

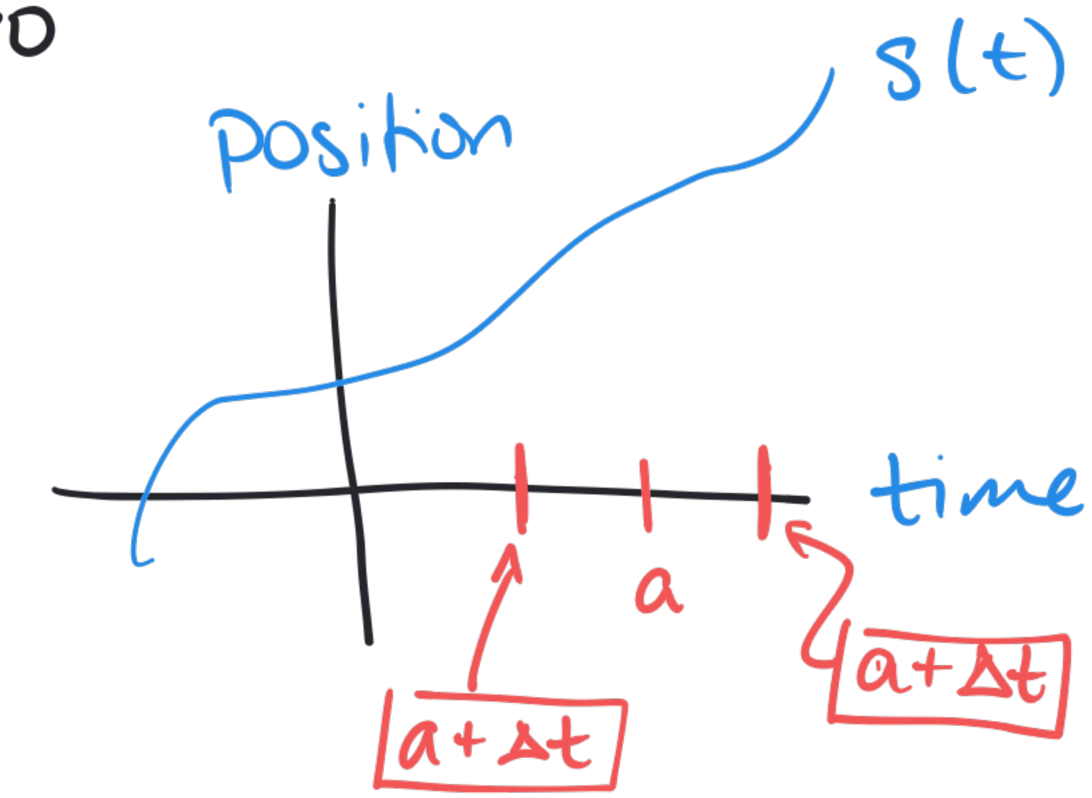


$\boxed{\text{VELOCITY}}$ The rate of change of distance with respect to time.

$$\text{Average velocity} = \frac{\Delta \text{ distance}}{\Delta \text{ time}}$$

$$\text{instantaneous velocity} = \lim_{\Delta t \to 0} \text{position}(t)$$

$$= \lim_{\Delta t \to 0} \frac{s(a+\Delta t) - s(a)}{\Delta t}$$



The instantaneous velocity at a of a position function is the slope of the tangent line to the graph of position at $(a, s(a))$.

Example: A ball is thrown in the air and its height in feet is given by

$$s(t) = 40t - 16t^2.$$

- What is its velocity after 2 s? After 1 s?

$$\text{instantaneous velocity at } t=1 = \lim_{h \to 0} \frac{\boxed{s(1+h)} - \boxed{s(1)}}{h}$$

$$= \lim_{h \to 0} \frac{40(1+h) - 16(1+h)^2 - (40(1) - 16(1)^2)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(40 + 40h - 16(1^2 + 2h + h^2) - (40 - 16)\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\cancel{40} + 40h \cancel{- 16} - 32h + h^2 - \cancel{40} + \cancel{16}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(40\cancel{h} - 32\cancel{h} + h^{\cancel{2}}\right) = \lim_{h \to 0} 40 - 32 + h = 8 \text{ ft/s}.$$

Suppose $f(x)$ is some function. Then the average rate of change on $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Define the instantaneous rate of change to be

$$\lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If $x_2 = x_1 + h$

$$= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{x_1 + h - x_1} = \boxed{\lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} = f'(x_1)}$$

THE DEFINITION OF THE DERIVATIVE!

The derivative at a measures

1. The slope of the tangent line at $(a, f(a))$
2. Instantaneous velocity at time $t = a$
3. Instantaneous rate of change of a function.